JOURNAL OF APPROXIMATION THEORY 31, 383–402 (1981)

Complex Planar Splines*

GERHARD OPFER

Department of Mathematics, Indiana University, Bloomington, Indiana 47405, and Universität Hamburg, Institut für Angewandte Mathematik, Bundesstrasse 55, D-2000 Hamburg 13, West Germany

AND

MADAN L. PURI

Department of Mathematics, Indiana University, Bloomington, Indiana 47405

Communicated by Lothar Collatz

Received July 7, 1980

In contradistinction to the known theory on complex splines which are defined on the boundary of a region in $\mathbb{C}$, we define complex planar splines on a region itself as a complex valued continuous function which is defined piecewise on suitable meshes of that region. The main idea is to use nonholomorphic functions as pieces, since holomorphic pieces would lead to just one holomorphic function on the whole region. Some of the techniques used are available from the theory of finite elements. But we also consider new aspects, namely, mapping properties of a complex planar spline v and the difference $f - v$, where f is, in general, a holomorphic function. For triangular meshes, rectangular and parallelogrammatic meshes, and meshes on circular sectors, explicit expressions are provided; also properties of the newly introduced complex planar splines are studied.

1. INTRODUCTION

In this paper we are concerned with the approximation of complex valued functions by functions which we would like to call complex *planar* splines.

In the current literature (see e.g., Ahlberg [1], Ahlberg *et al.* [2–4], Atteia [7], and Schoenberg [12, 13]), complex splines are defined on the boundary of a given region and are then extended into the interior by Cauchy's integral

* Research supported by the Office of Naval Research under Contract N00014-77-C-0659. The U.S. Government's right to retain a nonexclusive royalty-free license in and to the copyright covering this paper, for governmental purposes, is acknowledged.

0021-9045/81/040383–20$02.00/0

formula. However, this extension process is not easy to execute numerically. Therefore, we offer another approach, which in spirit originates from the theory of finite elements. (For full information on this subject, see the references in Schwarz [15].) We subdivide a given region into *meshes* and define a complex valued function on that region piecewise. The functions defined on each individual mesh will be called *elements*. The vertices of the occurring meshes will be called *grid points* and the set of all grid points a *grid*.

The minimum requirement we impose is the continuity of that piecewise defined function. Any such piecewise defined complex function which is continuous will be called a *complex planar spline*.

The continuity already has a very serious implication. If we try to define a complex planar spline by holomorphic elements like polynomials, then by the well known identity theorem (e.g., Diederich and Remmert [9, p. 132, Theorem 60]) all the elements represent just one holomorphic function. The consequence is that it makes no sense to work with holomorphic elements. Therefore, we have to use nonholomorphic elements.

Very simple nonholomorphic functions are polynomials in the complex variable z and its complex conjugate $\bar{z}$. These functions have the form

$$p(z, \bar{z}) = \sum_{j,k=0}^{n} a_{jk} z^j \bar{z}^k, \qquad a_{jk} \in \mathbb{C}. \tag{1.1}$$

The number

$$\partial p = \max_{j,k=0,1,\ldots,n} \{j + k: a_{j,k} \neq 0\} \tag{1.2}$$

will be called the *degree* of p. If $\partial p = 1$ we shall say that p is *linear*; if $\partial p = 2$ we shall say that p is *quadratic*; and if $\partial p = 3, 4, 5$ we shall use the words *cubic*, *quartic*, and *quintic*, respectively.

If we use the representation

$$z = x + iy, \qquad \bar{z} = x - iy \tag{1.3}$$

then

$$x = \frac{1}{2}(z + \bar{z}); \qquad y = \frac{1}{2i}(z - \bar{z}), \tag{1.4}$$

which means that a function in the real variables x and y can be transformed into a function depending on the complex variables z and $\bar{z}$, and vice versa.

Thus, a function $u = u(z, \bar{z})$ may also be regarded as a function in x and y. If u is continuously differentiable with respect to x and y, then we have

$$u_x = u_z + u_{\bar{z}}; \qquad u_y = i(u_z - u_{\bar{z}}), \tag{1.5}$$

$$u_z = \frac{u_x - iu_y}{2}; \qquad u_{\bar{z}} = \frac{u_x + iu_y}{2}. \tag{1.6}$$

If u is twice continuously differentiable, then

$$\Delta u = u_{xx} + u_{yy} = 4u_{z\bar{z}}. \tag{1.7}$$

In the sequel we treat triangular, rectangular and parallelogrammatic meshes, and meshes on circular sectors. The elements defined on those meshes will be as simple as possible. Besides the aspects known from the theory of finite elements, (for instance, interpolating properties, computational aspects, and an error analysis), there are new aspects which can be summarized by the term *mapping properties* of the newly defined complex planar splines. These new aspects concern the following questions, among others:

(I) How close are complex planar splines to comformality?

(II) Are complex planar splines quasiconformal?

(III) Are complex planar splines open mappings?

(IV) Is the boundary maximum principle valid for complex planar splines?

In some cases these questions apply also to the difference $\varepsilon = f - v$ between a certain function f and a complex planar spline v.

Since the interpolating formulae for the complex case appear different from the corresponding formulae for the real case, we believe that it is reasonable to state these formulae here. We shall see that the complex interpolation and L_2-approximation problem reduce to two real problems such that no new error analysis is needed. But the situation changes, for instance, for the uniform approximation problem.

Besides the approximation of functions by complex planar splines, still another application seems possible, namely, solving complex differential equations like Beltrami's equation

$$f_{\bar{z}} = \mu f_z + \nu \overline{f_z} \tag{1.8}$$

without splitting them into real and imaginary parts. Equation (1.8) was recently treated numerically by Weisel [16] for the case $\mu = 0$ by solving the corresponding real system with finite element techniques. A systematic treatment of (1.8) can be found in Wendland [17]. Furthermore, for the conformal mapping problem there also exists an investigation by Bosshard [8] on the use of finite elements, again by treating corresponding real

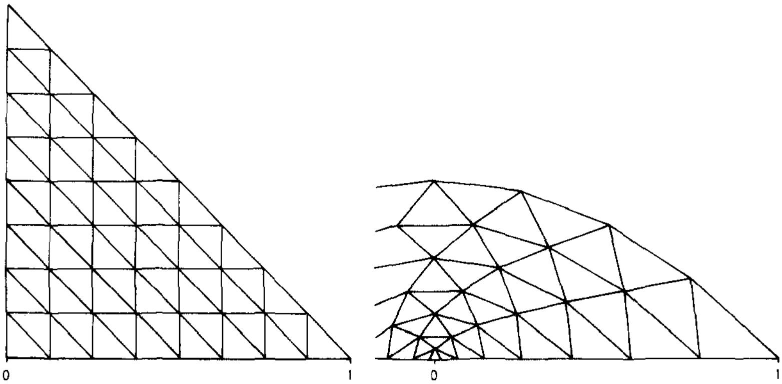

FIG. 1. Complex planar spline interpolating z^2, domain and range.

problems. Also in this case a direct attack seems possible, which then avoids the computation of the conjugate function to the computed real part of the mapping.

Still another application is the automatic construction of meshes with desired behavior. In many cases, for instance, it is desirable to construct meshes which concentrate at a certain point. An example is shown in Fig. 1, where the range of an interpolating complex planar spline is sketched. These applications provide motivation for our study of complex planar splines and their basic properties.

2. Linear Complex Planar Splines on Triangular Meshes

A *triangle* Δ is defined by its three vertices P_1, P_2, P_3 which are supposed to be three pairwise different complex numbers not located on a straight line. The three edges of Δ will be designated by $\overline{P_1P_2}$, $\overline{P_2P_3}$, $\overline{P_3P_1}$, where the order of the indices is irrelevant.

We first investigate one single element on a triangle Δ. We will use a linear element of the form

$$p(z, \bar{z}) = a + bz + c\bar{z}; \qquad a, b, c \in \mathbb{C}. \tag{2.1}$$

For simplicity of notation we shall write $p(z)$ instead of $p(z, \bar{z})$.

It is clear that the interpolation problem $p(P_j) = \zeta_j$, $j = 1, 2, 3$, has a unique solution for any three complex numbers $\zeta_1, \zeta_2, \zeta_3$. This solution is given by

$$a = (\zeta_1(P_2\overline{P_3} - \overline{P_2}P_3) + \zeta_2(\overline{P_1}P_3 - P_1\overline{P_3}) + \zeta_3(P_1\overline{P_2} - \overline{P_1}P_2))/\delta, \tag{2.2}$$

$$b = (\zeta_1(\overline{P_2} - \overline{P_3}) + \zeta_2(\overline{P_3} - \overline{P_1}) + \zeta_3(\overline{P_1} - \overline{P_2}))/\delta, \tag{2.3}$$

$$c = (\zeta_1(P_3 - P_2) + \zeta_2(P_1 - P_3) + \zeta_3(P_2 - P_1))/\delta, \tag{2.4}$$

where

$$\delta = 2i\,\mathrm{Im}(P_1\overline{P_2} + P_2\overline{P_3} + P_3\overline{P_1}). \tag{2.5}$$

(Note that $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ mean the real and imaginary parts of z, respectively.)

It is easy to see that any three points $z_1, z_2, z_3 \in \mathbb{C}$ form a triangle if and only if

$$\mathrm{Im}(z_1\overline{z_2} + z_2\overline{z_3} + z_3\overline{z_1}) \neq 0 \tag{2.6}$$

and this property is invariant under translation.

For the *standard triangle* $P_1 = 0$, $P_2 = 1$, $P_3 = i$ the above formulae reduce to

$$a = \zeta_1, \tag{2.7}$$

$$b = \tfrac{1}{2}(\zeta_1(i - 1) + \zeta_2 - i\zeta_3), \tag{2.8}$$

$$c = \tfrac{1}{2}(-\zeta_1(i + 1) + \zeta_2 + i\zeta_3), \tag{2.9}$$

with

$$\delta = -2i. \tag{2.10}$$

For any given triangle with vertices P_1, P_2, P_3 we can construct three special elements p_j, $j = 1, 2, 3$, by solving $p_j(P_k) = \delta_{jk}$, $j, k = 1, 2, 3$, where δ_{jk} is the common Kronecker symbol. These elements are usually called *form elements*. They have the property that the general interpolating element p with $p(P_j) = \zeta_j$, $j = 1, 2, 3$, can be written in the simple form

$$p = \sum_{j=1}^{3} \zeta_j p_j. \tag{2.11}$$

Their importance lies in the fact that they can be used to construct a basis for the linear space of all complex planar splines, as will be seen later.

DEFINITION 2.1. A linear element $p = a + bz + c\bar{z}$ for any $a, b, c \in \mathbb{C}$ is called *degenerate* if $|b| = |c|$.

Let R be a region in $\mathbb{C}$ and $f: R \to \mathbb{C}$ a mapping which has continuous partial derivatives f_z and $f_{\bar{z}}$. If f is quasiconformal, the numbers

$$d(z) = \frac{|f_z(z)| + |f_{\bar{z}}(z)|}{|f_z(z)| - |f_{\bar{z}}(z)|} \tag{2.12}$$

and

$$\mu(z) = \frac{f_{\bar{z}}(z)}{f_z(z)} \tag{2.13}$$

are called the *dilatation quotient* and *complex dilatation* at the point z, respectively (Lehto and Virtanen [10, p. 52, 191]). We always have $d(z) \geqslant 1$. In case $d(z) = 1$, z could be called a conformal point. If $d(z) = 1$ for all z, then f is conformal.

The following theorem gives some properties of a linear element p.

THEOREM 2.1. *A linear element* $p(z) = a + bz + c\bar{z}$, $a, b, c \in \mathbb{C}$ *has the following properties*:

1. *If* $z_1, z_2 \in \mathbb{C}$, *then*

$$p((1-\lambda) z_1 + \lambda z_2) = (1-\lambda) p(z_1) + \lambda p(z_2) \qquad \textit{for all } \lambda \in \mathbb{R}. \tag{2.14}$$

2. *The range of p applied to a triangle is again a triangle if and only if p is nondegenerate.*

3. *p is an orientation preserving homeomorphism if and only if* $|b| > |c|$.

4. *p is an open mapping if and only if p is nondegenerate.*

5. *p is quasiconformal if and only if* $|b| > |c|$. *In that case its dilatation is a constant given by*

$$d = \frac{|b| + |c|}{|b| - |c|} \tag{2.15}$$

and the complex dilatation is a constant given by

$$\mu = \frac{c}{b}. \tag{2.16}$$

(6) *If we apply p to an angle* τ, $0 < \tau < \pi$, *the angle* τ *is distorted by the angle*

$$\tilde{\tau} = \arctan \frac{|c|^2 + \operatorname{Re}(b\bar{c}) - \operatorname{Im}(b\bar{c}) \tan \tau}{\operatorname{Im}(b\bar{c}) + \frac{1}{2}(|c|^2 - |b|^2) \tan \tau - \frac{1}{2} |b+c|^2 \cot \tau}. \tag{2.17}$$

Proof. Properties 1–5 are immediate. The proof of property 6 follows by standard computations. ∎

We now come to another important application of Theorem 2.1.

DEFINITION 2.2. Two triangles Δ_1, Δ_2 are called *neighbors* or *neighboring* if they share a common edge.

THEOREM 2.2. *Let Δ_1, Δ_2 be two neighboring triangles with the common edge $\overline{P_1P_2}$ (see Fig. 2). Further let p_j be a linear element in Δ_j, $j = 1, 2$. If*

$$p_1(P_j) = p_2(P_j), \qquad j = 1, 2, \tag{2.18}$$

then $p_1(z) = p_2(z)$ for all $z \in \overline{P_1P_2}$, and consequently

$$\begin{aligned} p(z) &= p_1(z) \qquad \text{for} \quad z \in \Delta_1 \\ &= p_2(z) \qquad \text{for} \quad z \in \Delta_2 \end{aligned} \tag{2.19}$$

is continuous on $\Delta_1 \cup \Delta_2$.

Proof. Follows directly from Theorem 2.1 (property 1). ∎

Now if we subdivide a region into triangular meshes and define a linear element in each mesh we obtain a complex planar spline if we impose condition (2.18) for each pair of neighboring triangles. In the triangulation, however, we do not allow that a vertex of any triangle is interior to any other edge. More specifically the triangulation has to be *proper* (Prenter [11, p. 127]).

The form elements introduced earlier are used to construct a basis for the linear space V of all complex planar splines. Let R be a region in $\mathbb{C}$, subdivided into finitely many triangles and $P_1, P_2, \ldots, P_N$ its grid points. A complex planar spline associated with the grid point P_j and defined by

$$\begin{aligned} v_j(z) &= 1 \qquad \text{for} \quad z = P_j \\ &= 0 \qquad \text{for} \quad z \in \{P_1, P_2, \ldots, P_N\} - \{P_j\}, \quad j = 1, \ldots, N, \end{aligned}$$

will be called a *(global) form function*. It can be constructed piecewise from the (local) form elements already known. Assume that $T_1, T_2, \ldots, T_{k_j}$ are the

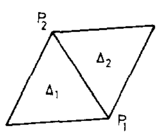

FIG. 2. Neighboring triangles.

triangles which have P_j as a common vertex. Then combine those form elements defined in $T_1, T_2, \ldots, T_{k_j}$ which have the value one at P_j and zero at the two other vertices with the zero elements on all other triangles. Clearly $V = \langle v_1, v_2, \ldots, v_N \rangle$.

A complex planar spline in general does not represent an open mapping even if all elements are not degenerate. To see this we define a complex planar spline on the square

$$Q = \{z: 0 \leqslant \operatorname{Re} z \leqslant 1, 0 \leqslant \operatorname{Im} z \leqslant 1\}, \tag{2.20}$$

which we divide into two triangles,

$$\Delta_1 = \{z \in Q: \operatorname{Re} z + \operatorname{Im} z \leqslant 1\}, \tag{2.21}$$

$$\Delta_2 = \{z \in Q: \operatorname{Re} z + \operatorname{Im} z \geqslant 1\}, \tag{2.22}$$

and define

$$\begin{aligned} v &= z && \text{for} \quad z \in \Delta_1 \\ &= 1 + i - i\bar{z} && \text{for} \quad z \in \Delta_2, \end{aligned} \tag{2.23}$$

which is then a complex planar spline on Q.

However, the range of Q under v is Δ_1 such that the range of any open set in Q containing parts of the diagonal of Q is not open.

But we have the following lemma (where the proof is obvious).

LEMMA 2.1. *If a complex planar spline represents a univalent function it is an open mapping.*

A consequence of this lemma is the following.

THEOREM 2.3. *Let v be a complex planar spline which interpolates a univalent, holomorphic mapping f on the grid points of a sufficiently fine triangular grid which is inside of the domain of definition of f. Then v is univalent and open.*

Proof. Since f is conformal, a small triangle is mapped such that the images of the three vertices form a triangle. Therefore, the interpolating complex planar spline maps the triangular grid onto another triangular grid in a univalent way. The result follows from Lemma 2.1. ∎

LEMMA 2.2. *Let f be a holomorphic function in a region R and g a holomorphic function in $\tilde{R} = \{z: \bar{z} \in R\}$. Define a function h on R by $h(z) =$*

$f(z)+g(\bar{z})$, $z \in R$. If h is not constant, then $|h|$ does not admit a maximum in R.

Proof. It is sufficient to show that $|h|^2$ has no maximum in R. If Δ represents the Laplace operator as defined in (1.7) we obtain

$$\Delta |h|^2 = 4\,|h|^2_{z\bar{z}} = 4(|f_z|^2 + |g_{\bar{z}}|^2) \geqslant 0.$$

But this implies that $|h|^2$ is subharmonic. Since h is not constant, $|h|^2$ does not admit a maximum in R (Ahlfors [6, p. 245]). ∎

Particularly a nonconstant linear element p itself has the property that it admits no maximum in the interior of the triangle in which it is defined. This is a stronger property than property 4 of Theorem 2.1.

Clearly, a complex planar spline is not subharmonic in general. As an example take an interpolating spline which is 1 at one interior grid point and 0 at the other grid points.

3. Quadratic Complex Planar Splines on Rectangular and Parallelogram Meshes

First we study splines on rectangles whose sides are parallel to the axes. Such a rectangle □ (called *quabla* in physics) is defined by its four vertices P_1, P_2, P_3, P_4, which are to be understood as pairwise distinct complex numbers with $\operatorname{Im} P_1 = \operatorname{Im} P_2$, $\operatorname{Im} P_3 = \operatorname{Im} P_4$, $\operatorname{Re} P_1 = \operatorname{Re} P_4$, $\operatorname{Re} P_2 = \operatorname{Re} P_3$ arranged in positive orientation.

We first investigate one single element on a rectangle □. A quadratic element of the form

$$p(z) = a + bz + c\bar{z} + d(z^2 - \bar{z}^2), \qquad a, b, c, d \in \mathbb{C}, \tag{3.1}$$

will be used. That the use of this element is reasonable will be seen in Theorem 3.1. The interpolation problem $p(P_j) = \zeta_j$, $j = 1, 2, 3, 4$, again has a unique solution for any four complex numbers $\zeta_1, \zeta_2, \zeta_3, \zeta_4$:

$$\begin{aligned} a = (1/\delta)((P_3^2 - \bar{P}_3^2)\,\zeta_1 - (P_4^2 - \bar{P}_4^2)\,\zeta_2 \\ + (P_1^2 - \bar{P}_1^2)\,\zeta_3 - (P_2^2 - \bar{P}_2^2)\,\zeta_4), \end{aligned} \tag{3.2}$$

$$b = (2/\delta)(-P_3\zeta_1 + P_4\zeta_2 - P_1\zeta_3 + P_2\zeta_4), \tag{3.3}$$

$$c = (2/\delta)(\bar{P}_3\zeta_1 - \bar{P}_4\zeta_2 + \bar{P}_1\zeta_3 - \bar{P}_2\zeta_4), \tag{3.4}$$

$$d = (1/\delta)(\zeta_1 - \zeta_2 + \zeta_3 - \zeta_4), \tag{3.5}$$

where

$$\delta = 4(P_2 - P_1)(P_3 - P_2). \tag{3.6}$$

In the case of quadratic complex planar splines on rectangular meshes, we have the following analogue of Theorem 2.1.

THEOREM 3.1. *Let P, Q any two distinct points of $\mathbb{C}$ and p a quadratic element as defined in* (3.1) *with $d \neq 0$.*
Then

$$p((1-\lambda)P + \lambda Q) = (1-\lambda)\,p(P) + \lambda p(Q) \qquad \text{for all } \lambda \in [0, 1] \tag{3.7}$$

if and only if the straight line through P and Q is parallel to the x- or y-axis.

Proof. Set $z_\lambda = (1-\lambda)P + \lambda Q$. Then because of Theorem 2.1 and $d \neq 0$, (3.7) is equivalent to

$$z_\lambda^2 - \bar{z}_\lambda^2 = (1-\lambda)(P^2 - \bar{P}^2) + \lambda(Q^2 - \bar{Q}^2) \qquad \text{for all } \lambda \in [0, 1]. \tag{3.8}$$

Let $P = x + iy$ and $Q = u + iv$. After routine computations we deduce from (3.8) that $x(v-y) = u(v-y)$, which yields the assertion. ∎

DEFINITION 3.1. Two rectangles will be called *neighbors* or *neighboring* if they share a common edge (Fig. 3).

THEOREM 3.2. *Let $\square_1, \square_2$ be two neighboring rectangles whose sides are parallel to the axes, with the common edge $\overline{P_1P_2}$ (Fig. 3). Let p_j be a quadratic element of the form* (3.1) *defined on $\square_j$, $j = 1, 2$. If*

$$p_1(P_j) = p_2(P_j), \qquad j = 1, 2, \tag{3.9}$$

then

$$p_1(z) = p_2(z) \qquad \text{for all } z \in \overline{P_1P_2}.$$

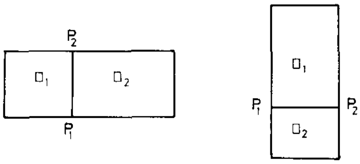

FIG. 3. Neighboring rectangles.

Consequently

$$\begin{aligned} p(z) &= p_1(z) \quad \text{for} \quad z \in \Box_1 \\ &= p_2(z) \quad \text{for} \quad z \in \Box_2 \end{aligned} \tag{3.10}$$

is continuous on $\Box_1 \cup \Box_2$.

Proof. Follows directly from Theorem 3.1. ▮

Now if we subdivide a region into rectangular meshes where the sides of the rectangles have to be parallel to the axes then we obtain a complex planar spline if we impose condition (3.9) for each pair of neighboring rectangles.

THEOREM 3.3. *Let R be any region in $\mathbb{C}$, f a holomorphic function in R and p a quadratic element as defined in* (3.1). *If $f-p$ is not constant, then $|f-p|$ does not admit a maximum in R.*

Proof. Since $f-p$ is a sum of a holomorphic function in z and a holomorphic function in $\bar{z}$, Lemma 2.2 applies and yields the desired result. ▮

This theorem applies particularly to p itself.

More information about the quadratic element p as defined in (3.1) can be deduced from its Jacobi determinant

$$J = |p_z|^2 - |p_{\bar{z}}|^2 \tag{3.11}$$

(see Lehto and Virtanen [10, p. 136]), which reads, in the present case,

$$\begin{aligned} J(z) &= |b + 2dz|^2 - |c - 2d\bar{z}|^2 \\ &= |b|^2 - |c|^2 + 2(d\bar{b} + \bar{d}c)z + 2(\bar{d}b + d\bar{c})\bar{z}. \end{aligned} \tag{3.12}$$

The set

$$H = \{z : J(z) = 0\} \tag{3.13}$$

is either a straight line or a point in $\mathbb{C}$.

DEFINITION 3.2. The quadratic element $p(z) = a + bz + c\bar{z} + d(z^2 - \bar{z}^2)$, $a, b, c, d, z \in \mathbb{C}$, is called *degenerate* if

$$\sigma = d\bar{b} + \bar{d}c = 0. \tag{3.14}$$

Now, H of (3.13) is a straight line in $\mathbb{C}$ if and only if p is not degenerate. If p is degenerate then (3.14) implies $d = 0$ or $|b| = |c|$. If p is not degenerate then the straight line H forms the angle α with the real axis, which is given by

$$\tan\alpha = \frac{\operatorname{Re}(d\bar{b} + \bar{d}c)}{\operatorname{Im}(d\bar{b} + \bar{d}c)}, \qquad 0 \leqslant \alpha < \pi. \tag{3.15}$$

Let us assume now that p is not degenerate. If the domain of definition of p is any compact set S in $\mathbb{C}$ which is located in the half plane

$$H^+ = \{z : J(z) > 0\} \tag{3.16}$$

then p is locally an orientation preserving homeomorphism in S which then is also quasiconformal in S since its dilatation quotient (see 2.12) is bounded.

In order to find out whether p is a global homeomorphism we study the solutions of

$$p(z_2) - p(z_1) = 0. \tag{3.17}$$

If we use the abbreviations

$$x = z_2 - z_1, \qquad y = z_2 + z_1 \tag{3.18}$$

Eq. (3.17) reads

$$p(z_2) - p(z_1) = x(b + dy) + \bar{x}(c - d\bar{y}) = 0. \tag{3.19}$$

From this it follows that

$$d\overline{(p(z_2) - p(z_1))} + \bar{d}(p(z_2) - p(z_1)) = x\bar{\sigma} + \bar{x}\sigma = 0, \tag{3.20}$$

where σ was already introduced in (3.14). If we use this equation to eliminate $\bar{x}$ from (3.19) we obtain

$$y\sigma + \bar{y}\bar{\sigma} + |b|^2 - |c|^2 = 0 \tag{3.21}$$

in case $x \neq 0$. Let $J(z_j) > 0$, $j = 1, 2$. Then from (3.12) by forming $J(z_1) + J(z_2)$ it follows that

$$2(|b|^2 - |c|^2) + 2\sigma y + 2\bar{\sigma}\bar{y} > 0, \tag{3.22}$$

which contradicts (3.21).

To summarize: If p is not degenerate then it is univalent in both half planes H^+ and $H^- = \{z : J(z) < 0\}$.

Since the domain of definition of p is a rectangle R whose sides are parallel to the axes, one can find out whether $R \subset H^+$ just by inserting a suitable vertex into J. To explain that we distinguish four cases according to the special location of H^+ in $\mathbb{C}$.

Case 1. $\pi/2 \leqslant \alpha < \pi$ and H^+ is a right half plane in $\mathbb{C}$.

Case 2. $\alpha = 0$ and H^+ is an upper half plane in $\mathbb{C}$, or $0 < \alpha < \pi/2$ and H^+ is a left half plane in $\mathbb{C}$.

Case 3. $\pi/2 \leqslant \alpha < \pi$ and H^+ is a left half plane in $\mathbb{C}$.

Case 4. $\alpha = 0$ and H^+ is a lower half plane in $\mathbb{C}$, or $0 < \alpha < \pi/2$ and H^+ is a right half plane in $\mathbb{C}$.

The words *left*, *right*, *lower*, *upper* half plane are used in the ordinary sense.

THEOREM 3.4. *Let R be a rectangle whose sides are parallel to the axes. Call P_1 the lower left, P_2 the lower right, P_3 the upper right and P_4 the upper left vertex of R. Further let p be a nondegenerate quadratic element (as defined in* (3.1)) *on R. Then the mapping p is an orientation preserving homeomorphism and quasiconformal on R if and only if $J(P_j) > 0$, where j is determined by the case number j to which H^+ belongs, $j \in \{1, 2, 3, 4\}$.*

Proof. If we are in case j, $j \in \{1, 2, 3, 4\}$, then $J(P_j) > 0$ is equivalent to $R \subset H^+$. ∎

If for mnemonic reasons one would like to give this theorem a name, then *four corner theorem* seems to be very suitable, since all four corners of the rectangle R are involved.

Under the assumptions stated p will be an orientation *inverting* homeomorphism on R if and only if $J(P_k) < 0$, where $k = (j + 2) \bmod 4$ and j is determined as before.

If a rectangle is subdivided in this way into $m \cdot n$ little rectangles, then there are $m \cdot n$ elements having $p = 4m \cdot n$ parameters altogether. Further there are $s = 3mn - m - n - 1$ continuity conditions leaving $p - s = (m + 1)(n + 1)$ parameters free, where $(m + 1)(n + 1)$ is also the number of grid points.

Now we can adjust the element (3.1) to the case where the rectangle has any position in the plane. If one of the edges of a given rectangle forms the angle α with the x-axis then instead of (3.1) one must use

$$p(z) = a + bz + c\bar{z} + d(z^2 - e^{4i\alpha}\bar{z}^2). \tag{3.23}$$

Since a parallelogram can be mapped by a linear transformation l of the type (2.1) onto a rectangle whose sides are parallel to the axes one can also work with parallelograms.

If $\alpha_1, \alpha_2 \in \mathbb{C}$, with $\alpha_1 \neq \lambda\alpha_2$ for any $\lambda \in \mathbb{R}$, describe the directions of the parallelogram grid, which means that the two angles β_j, $j = 1, 2$, with the real axis are given by $\tan \beta_j = \operatorname{Re} \alpha_j / \operatorname{Im} \alpha_j$, $j = 1, 2$, then the aforementioned linear transformation l yields an element of the form

$$p(z) = a + bz + c\bar{z} + d(\alpha_1 z + \overline{\alpha_1 z})(\alpha_2 z + \overline{\alpha_2 z}), \quad a, b, c, d \in \mathbb{C}. \tag{3.24}$$

4. Splines on Disks and Circular Sectors

Let S be a circular sector. If we divide the radius into k subintervals and the opening angle into $l - 1$ subintervals, we obtain, in total, $k(l-1)$ meshes of two types, which we would call *rectangular* and *triangular* meshes, respectively. The triangular meshes contain the origin O of S, whereas the rectangular meshes do not contain the origin (see Fig. 4).

It is straightforward to use polar coordinates with respect to the origin O of S in this situation. Therefore for rectangular meshes we use an element of the form

$$p(z) = p(re^{i\phi}) = a + br + c\phi + dr\phi, \qquad a, b, c, d \in \mathbb{C}. \tag{4.1}$$

Let P_1, P_2, P_3, P_4 be the vertices of one specific rectangular mesh in positive orientation such that $\phi_1 = \arg P_1 = \arg P_2$, $\phi_2 = \arg P_3 = \arg P_4$, $r_1 = |P_1| = |P_4|$, $r_2 = |P_2| = |P_3|$ (see Fig. 5).

Then the solution of the interpolation problem

$$p(P_j) = \zeta_j, \qquad \zeta_j \in \mathbb{C}, \quad j = 1, 2, 3, 4, \tag{4.2}$$

is given by

$$a = (r_2\phi_2\zeta_1 - r_1\phi_2\zeta_2 + r_1\phi_1\zeta_3 - r_2\phi_1\zeta_4)/\delta, \tag{4.3}$$

$$b = (-\phi_2\zeta_1 + \phi_2\zeta_2 - \phi_1\zeta_3 + \phi_1\zeta_4)/\delta, \tag{4.4}$$

$$c = (-r_2\zeta_1 + r_1\zeta_2 - r_1\zeta_3 + r_2\zeta_4)/\delta, \tag{4.5}$$

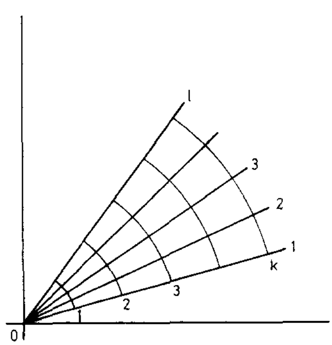

FIG. 4. Sector S subdivided into rectangular and triangular meshes.

and

$$d = (\zeta_1 - \zeta_2 + \zeta_3 - \zeta_4)/\delta, \tag{4.6}$$

where

$$\delta = 1/(r_2 - r_1)(\phi_2 - \phi_1). \tag{4.7}$$

Now assume that O, P, Q are the vertices of a triangular mesh in positive orientation, where O is the origin of the sector S (see Fig. 5). In order to find out what type of element to use, we study the interpolation problem for a rectangular mesh, where $P_1 \to 0$, $P_4 \to 0$ and $\zeta_1 = \zeta_4$, such that $|P_1| = |P_4|$ and $\arg P_1 = \arg P_2$, $\arg P_4 = \arg P_3$.

After some computation we find $c \to 0$ in (4.1). This means that we have to use an element of the form

$$p(z) = \alpha + \beta r + \gamma r\phi, \qquad \alpha, \beta, \gamma \in \mathbb{C}. \tag{4.8}$$

If we assume that P and Q have the polar coordinates (r_1, ϕ_1) and (r_2, ϕ_2), respectively, then the interpolation problem

$$p(O) = \zeta_1, \qquad p(P) = \zeta_2, \qquad p(Q) = \zeta_3, \qquad \zeta_1, \zeta_2, \zeta_3 \in \mathbb{C}, \tag{4.9}$$

has the solution

$$\alpha = \zeta_1, \tag{4.10}$$

$$\beta = (-(\phi_2 - \phi_1)\,\zeta_1 + \phi_2\zeta_2 - \phi_1\zeta_3)/\delta, \tag{4.11}$$

and

$$\gamma = (-\zeta_2 + \zeta_3)/\delta, \tag{4.12}$$

where

$$\delta = 1/r_1(\phi_2 - \phi_1). \tag{4.13}$$

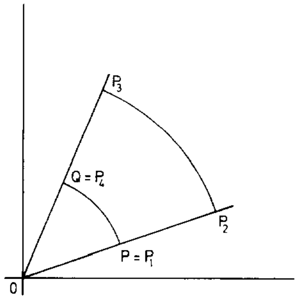

FIG. 5. Triangular and rectangular mesh of a circular sector.

THEOREM 4.1. *Let S be a circular sector subdivided into meshes as described above and m_j, $j=1,2$ two neighboring meshes sharing the common edge $E=\overline{P_1P_2}$. Let p_j be an element defined on m_j, where p_j has the form* (4.1) *if m_j is a rectangular mesh or the form* (4.8) *if m_j is a triangular mesh, $j=1,2$.*

If

$$p_1(P_j)=p_2(P_j), \qquad j=1,2, \tag{4.14}$$

then

$$p_1(z)=p_2(z) \qquad \text{for all } z\in E. \tag{4.15}$$

Consequently,

$$\begin{aligned} p(z) &= p_1(z) \qquad \text{for} \quad z\in m_1 \\ &= p_2(z) \qquad \text{for} \quad z\in m_2 \end{aligned} \tag{4.16}$$

is continuous on $m_1\cup m_2$.

Proof. The elements (4.1) on rectangular meshes as well as the elements (4.8) on triangular meshes are linear on the edges of their respective domain of definition. ∎

The element p introduced in (4.1) has the same form as the element p defined in (3.1) and used for ordinary rectangles.

In order to see this, one has only to identify $\operatorname{Re} z$ with r and $\operatorname{Im} z$ with ϕ. The consequence is that no particular analysis is required besides that for elements on ordinary rectangles.

We end this section with the computation of the number of free parameters in a complex planar spline on a circular sector.

The $k(l-1)$ meshes distribute in $(k-1)(l-1)$ rectangular and $l-1$ triangular meshes. In order to make a piecewise defined function continuous we have to impose $l(3k-1)-4k$ conditions. The number of parameters is $(l-1)(4k-1)$, leaving $lk+1$ parameters free. This is also the number of grid points.

5. LEAST-SQUARES APPROXIMATION AND INTERPOLATION WITH COMPLEX PLANAR SPLINES

If we want to approximate a complex valued function by complex planar splines of a certain type, we end up with minimizing a real valued functional defined in $\mathbb{C}^N$, $N\in\mathbb{N}$.

To treat such a problem it is not necessary to rewrite it in real form. Let us assume that we have to handle the problem

$$\phi(a) = \min, \qquad \phi\colon \mathbb{C}^N \to \mathbb{R}, \tag{5.1}$$

where ϕ may also explicitly depend on $\bar{a}$. If ϕ has continuous partial derivatives with respect to all components of a, then

$$\frac{\partial \phi(\hat{a})}{\partial a_j} = 0, \qquad j = 1, 2, ..., N, \quad a = (a_1, a_2, ..., a_n), \tag{5.2}$$

is a necessary condition for $\hat{a}$ to be a minimum of ϕ. This follows immediately from (1.5) and (1.6).

If $g, h\colon \mathbb{C}^N \to \mathbb{C}$ are complex valued functions possessing continuous partial derivatives with respect to a and $\bar{a}$, then

$$(\bar{g})_a = \overline{g_{\bar{a}}}, \qquad (\bar{g})_{\bar{a}} = \overline{g_a}, \tag{5.3}$$

$$(gh)_a = g_a h + \overline{g h_a}, \tag{5.4}$$

$$\frac{\partial}{\partial a} |g|^2 = \frac{\partial}{\partial a} (g\bar{g}) = g_a \bar{g} + g\overline{g_{\bar{a}}} = \overline{\frac{\partial}{\partial \bar{a}} |g|^2}, \tag{5.5}$$

and if g does not depend on $\bar{a}$ explicitly, then

$$\frac{\partial}{\partial a} |g|^2 = g_a \bar{g}. \tag{5.6}$$

The least-squares problem can be treated along the lines of Schultz [14, Chap. 6] as follows.

Let R be a compact set in $\mathbb{C}$ subdividable into meshes of the discussed form, $f \in L_2(R)$ and V the linear space of all complex planar splines, where $V = \langle v_1, v_2, ..., v_N \rangle$ and the v_j are the global form functions defined earlier, $j = 1, 2, ..., N$. It should be noticed that the form functions are real by definition.

The problem here is to minimize

$$\phi(a) = \int_R \left| f(z) - \sum_{j=1}^{N} a_j v_j \right|^2 dx\, dy, \qquad a = (a_1, a_2, ..., a_N) \in \mathbb{C}^N. \tag{5.7}$$

Using (5.2) to (5.7), we obtain

$$\frac{\partial}{\partial a_j} \phi(a) = \int_R \left(\sum_{k=1}^{N} \bar{a}_k v_j v_k - v_j \bar{f} \right) dx\, dy = 0,$$

which reduces to the linear system

$$Ca = r, \tag{5.8}$$

where

$$C = (c_{jk}) = \int_R v_j v_k \, dx \, dy, \qquad j, k = 1, 2, \ldots, N, \tag{5.9}$$

$$r = (r_j) = \int_R f v_j \, dx \, dy, \qquad j = 1, 2, \ldots, N. \tag{5.10}$$

Since C is a real matrix, system (5.8) can be partitioned into the two real systems,

$$C \operatorname{Re} a = \operatorname{Re} r, \qquad C \operatorname{Im} a = \operatorname{Im} r, \tag{5.11}$$

where $\operatorname{Re} a$ is the vector of the real parts of a; analogous meanings apply to $\operatorname{Re} r$, $\operatorname{Im} a$, and $\operatorname{Im} r$.

Clearly (5.8) has a unique solution since ϕ is strictly convex.

Let $\hat{v}$ be the best least-squares approximation of f. The error analysis can be directly taken from the real case (e.g., Schultz [14, Chap. 6]) since

$$\begin{aligned} \|f - \hat{v}\| &= \|\operatorname{Re}(f - \hat{v}) + i \operatorname{Im}(f - \hat{v})\| \\ &\leqslant \|\operatorname{Re} f - \operatorname{Re} \hat{v}\| + \|\operatorname{Im} f - \operatorname{Im} \hat{v}\|. \end{aligned} \tag{5.12}$$

This means that the order of convergence is the same as that in the real case, but the convergence constants have to be doubled, provided, of course, that $\operatorname{Re} f$ and $\operatorname{Im} f$ are of the same smoothness.

An interpolation problem

$$v(z_k) = f(z_k), \qquad k = 1, 2, \ldots, K, \tag{5.13}$$

where v is a complex planar spline, f is a given function, and z_k, $k = 1, 2, \ldots, K$, are given grid points, may also be partitioned into two real problems by splitting Eq. (5.13) into real and imaginary parts. The above remarks on the error analysis therefore also apply here.

6. A Numerical Example

We subdivide the standard triangle 0, 1, i in the usual way by dividing its two smaller sides into $1/h = 2^k$, $k = 0, 1, \ldots, 5$, pieces of equal length and divided the resulting little squares of side length h diagonally by parallels through the hypotenuse of the standard triangle. We obtained a complex

TABLE I

Spline Interpolating the Exponential Function on Standard Triangle

h	e_h	c_h	d_h	$\tilde{c}_h$
1	0.3730	1.54	2.229	1.45
1/2	0.1285	1.79	1.451	1.19
1/4	0.0372	1.90	1.198	1.09
1/8	0.00996	1.95	1.093	1.05
1/16	0.00257	1.98	1.045	1.03
1/32	0.000653		1.022	

planar spline v_h by interpolating the exponential function at the grid points of that triangle.

In Table I we list the computed values $e_h = \|\exp - v_h\|_\infty$, the corresponding numerical convergence order c_h of e_h, the maximal dilatation quotient d_h over all meshes (compare Formula (2.15)), and the corresponding convergence order $\tilde{c}_h$ of d_h.

The fact that e_h approaches 0 with order 2 is of course known (Schultz [14, Chap. 2]). According to our computation the maximal dilatation approaches one with order one. The number $d_h - 1$ could be called *deviation* from conformality. To the best of our knowledge, neither numerical values for the dilatation d_h nor theoretical investigations on the behavior of d_h as $h \to 0$ appear in literature.

Acknowledgments

The assistance of Mr. Grothkopf and Mr. Zieher for preparing the figures and the table is acknowledged.

References

1. J. H. Ahlberg, Splines in the complex plane, *in* "Approximations with Special Emphasis on Spline Functions" (I. J. Schoenberg, Ed.), pp. 1–27, Academic Press, New York/London, 1969.
2. J. H. Ahlberg, E. N. Nilson, and J. L. Walsh, "The Theory of Splines and Their Applications," Academic Press, New York/London, 1967.
3. J. H. Ahlberg, E. N. Nilson, and J. L. Walsh, Complex cubic splines, *Trans. Amer. Math. Soc.* **129** (1967), 391–413.
4. J. H. Ahlberg, E. N. Nilson, and J. L. Walsh, Properties of analytic splines. I. Complex polynomial splines, *J. Math. Anal. Appl.* **27** (1969), 262–278.
5. J. H. Ahlberg, E. N. Nilson, and J. L. Walsh, Complex polynomial splines on the unit circle, *J. Math. Anal. Appl.* **33** (1971), 234–257.

6. L. AHLFORS, "Complex Analysis," 3rd ed., McGraw–Hill, New York, 1979.
7. M. ATTEIA, Fonctions ⟪spline⟫ dans le champ complexe, *C.R. Acad. Sci. Paris* **273** (1971), 678–681.
8. F. BOSSHARD, "Die Konstruktion konformer Abbildungen mit der Methode der finiten Elemente," Dissertation, Universität Zürich, 1980.
9. K. DIEDERICH AND R. REMMERT, "Funktionentheorie I," Springer-Verlag, Berlin/Heidelberg/New York, 1972.
10. O. LEHTO AND K. I. VIRTANEN, "Quasikonforme Abbildungen," Springer-Verlag, Berlin/Heidelberg/New York, 1965.
11. P. PRENTER, "Splines and Variational Methods," Wiley, New York, 1975.
12. I. J. SCHOENBERG, On polynomial spline functions on the circle. I. Their interpolatory properties, *in* "Proceedings, Conference on Constructive Theory of Functions, Budapest, 1969" (G. Alexits and S. B. Stechkin, Eds.), pp. 403–418, Akadémiai Kiadó, Budapest, 1972.
13. I. J. SCHOENBERG, On polynomial spline functions on the circle. II. Monosplines and quadrature formulae, *in* "Proceedings, Conference on Constructive Theory of Functions, Budapest, 1969" (G. Alexits and S. B. Stechkin, Eds.), pp. 419–433, Akadémiai Kiadó, Budapest, 1972.
14. M. H. SCHULTZ, "Spline Analysis," Prentice–Hall, Englewood Cliffs, N.J., 1973.
15. H. R. SCHWARZ, "Methode der finiten Elemente," Teubner, Stuttgart, 1980.
16. J. WEISEL, Numerische Ermitlung quasikonformer Abbildungen mit finiten Elementen, *Numer. Math.* **35** (1980), 201–222.
17. W. L. WENDLAND, "Elliptic Systems in the Plane," Pitman, San Francisco/Melbourne, 1979.